Can Modalities Save Naive Set Theory?

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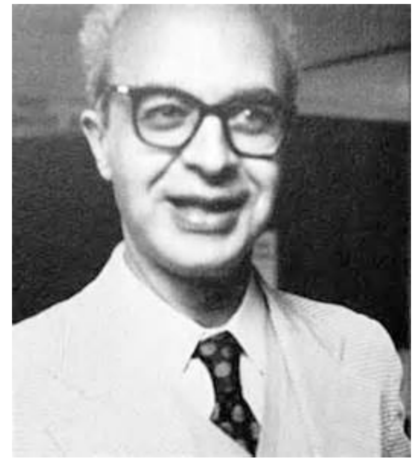
would never been possible without Dropbox!

Dedicated to the Memories of

Grigori “Grisha” Mints

Born: 7 June 1939 in St. Petersburg
Died: 29 May 2014 in Stanford



Georg Kreisel FRS

Born: 15 September 1923 in Graz
Died: 1 March 2015 in Salzburg

History

The late Grisha Mints asked Scott whether a naive set theory could be consistent in modal logic. Here are two modal forms of comprehension:

$$(\exists y)(\forall x)(x \in y \leftrightarrow \Box\varphi) \qquad (\mathrm{Comp}\Box)$$

$$(\exists y)(\forall x)\Box(x \in y \leftrightarrow \Box\varphi) \qquad (\Box\mathrm{Comp}\Box)$$

At the time (2009) neither he nor Scott knew the answer.

In the most commonly used systems, where the Converse Barcan Formula ($\Box\forall x\varphi \to \forall x\Box\varphi$) is derivable, ($\Box$Comp$\Box$) follows from another comprehension principle:

$$(\exists y)\Box(\forall x)(x \in y \leftrightarrow \Box\varphi)$$

In lectures Scott had presented a modal version of ZF which uses:

$$(\exists y)\Box(\forall x)(x \in y \leftrightarrow x \in u \wedge \varphi) \qquad (\mathrm{MZF\ Comp})$$

Background

Modalized comprehension principles have been studied in a number of different settings in the literature. Here are a few:

- intensional higher-order logic. see, e.g., Gallin (1975, p. 77) or Zalta (1988, p. 22)
- modalizing common set theories. Fine (1981); Shapiro (1985)
- making the iterative conception explicit. Studd (2013) and Linnebo (2013)
- comprehension with intensional biconditional: Aczel & Feferman (1980)
- Krajíček (1987), Krajíček (1988):

$$(\exists y)(\forall x)((\Box x \in y \leftrightarrow \Box\varphi) \wedge (\Box\neg x \in y \leftrightarrow \Box\neg\varphi)) \tag{MCA}$$

- Fitch (1966), Fitch (1967b), Fitch (1967a)

The Logical Setup

Our language will be the language of predicate logic with $\neg, \wedge, \forall, \exists$, plus identity $=$ and the relation symbol $\in$, along with the unary modal operator $\Box$. The modal system **T** can then be axiomatized with the following schematic axioms and rules:

(LPC)	Any substitution instance of a theorem of predicate logic
(K)	$\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
(T)	$\vdash \Box\varphi \to \varphi$
(MP)	From $\vdash \varphi$ and $\vdash \varphi \to \psi$ infer $\vdash \psi$
(UG)	From $\vdash \varphi \to \psi$ infer $\vdash \varphi \to \forall x\psi$, if x is not free in φ
(RN)	From $\vdash \varphi$ infer $\vdash \Box\varphi$

Using (K), (MP), and (RN), one shows that **T** has the derived rule:

$$\text{(RM)} \quad \text{From } \vdash \varphi \to \psi \text{ infer } \vdash \Box\varphi \to \Box\psi$$

There is no derivation of the Barcan formula ($\forall x\Box\varphi \to \Box\forall x\varphi$). However, the converse Barcan formula ($\Box\forall x\varphi \to \forall x\Box\varphi$) holds.

The Inconsistency of ($\Box$Comp$\Box$)

We can employ the following single instance of ($\Box$Comp$\Box$):

$$(\exists y)(\forall x)\Box(x \in y \leftrightarrow \Box\neg x \in x) \qquad (\Box\text{Russell}\Box)$$

Theorem. ($\Box$Russell$\Box$) *is inconsistent in* **T**.

Proof.

(1)	$(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x) \to \Box(R \in R \leftrightarrow \Box\neg R \in R)$	UI
(2)	$\Box(R \in R \leftrightarrow \Box\neg R \in R) \to (R \in R \leftrightarrow \Box\neg R \in R)$	(T)
(3)	$(R \in R \leftrightarrow \Box\neg R \in R) \to (R \in R \to \neg R \in R)$	(T)
(4)	$(R \in R \leftrightarrow \Box\neg R \in R) \to \neg R \in R$	3, PL
(5)	$\Box(R \in R \leftrightarrow \Box\neg R \in R) \to \Box\neg R \in R$	4, RM
(6)	$(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x) \to (R \in R \leftrightarrow \Box\neg R \in R)$	2, 3
(7)	$(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x) \to$ $(\Box\neg R \in R \wedge (R \in R \leftrightarrow \Box\neg R \in R))$	1, 5, 6, PL
(8)	$(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x) \to (\Box\neg R \in R \wedge R \in R)$	7, PL
(9)	$(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x) \to (\neg R \in R \wedge R \in R)$	8, (T), PL
(10)	$\neg(\forall x)\Box(x \in R \leftrightarrow \Box\neg x \in x)$	9, PL
(11)	$\neg\exists y\forall x\Box(x \in y \leftrightarrow \Box\neg x \in x)$	10, UG, PL

Proving Inconsistency of ($\Box$Comp$\Box$)

Note that this result does not depend on special laws for the quantifier, as it may be seen as an instance of the following general fact about the propositional fragment of the logic (exercise, using the model of the previous proof!):

Proposition. *If* $\vdash \varphi \to (\psi \leftrightarrow \Box\neg\psi)$, *then* $\vdash \neg\Box\varphi$ *in* **T**.

Letting $(R \in R \leftrightarrow \Box\neg R \in R) = \varphi$ and $R \in R = \psi$, we then derive the contradiction for ($\Box$Comp$\Box$) as follows:

$$
\begin{array}{lll}
(1) & \neg\Box(R \in R \leftrightarrow \Box\neg R \in R) & \\
(2) & \exists x\neg\Box(x \in R \leftrightarrow \Box\neg x \in x) & 1, \text{EI} \\
(3) & \neg\forall x\Box(x \in R \leftrightarrow \Box\neg x \in x) & 1, 2, \text{PL} \\
(4) & \forall y\neg\forall x\Box(x \in y \leftrightarrow \Box\neg x \in x) & 3, \text{UG} \\
(5) & \neg\exists y\forall x\Box(x \in y \leftrightarrow \Box\neg x \in x) & 4, \text{PL}
\end{array}
$$

The Inconsistency of ($\Box$Comp$\Box\Diamond$)

Having seen that the simple version of ($\Box$Comp$\Box$) is inconsistent in **T**, we can ask whether deeper modalities might be helpful. In doing so, it is better to move to the Lewis system **S4** where there are fewer compositions of modal operators. And we need to check as before:

Proposition. If $\vdash \Box\varphi \to (\psi \leftrightarrow \Box\neg\psi)$ then $\vdash \neg\Box\varphi$ in **S4**.

In the following variant comprehension principle, $\Diamond$ stands for $\neg\Box\neg$.

$$(\exists y)(\forall x)\Box(x \in y \leftrightarrow \Box\Diamond\varphi) \tag{$\Box$Comp$\Box\Diamond$}$$

However, this new principle is also inconsistent, as shown by:

$$(\exists y)(\forall x)\Box(x \in y \leftrightarrow \Box\Diamond\neg x \in x) \tag{$\Box$Russell$\Box\Diamond$}$$

Theorem. ($\Box$Russell$\Box\Diamond$) is inconsistent in **S4**.

Indeed this can be reduced back to the previous case.

Proving the Inconsistency of ($\Box$Comp$\Box\Diamond$)

Proposition. If $\vdash \Box\varphi \to (\psi \leftrightarrow \Box\Diamond\neg\psi)$ then $\vdash \neg\Box\varphi$ in **S4**.

Proof:

(1)	$\Box\varphi \to (\psi \leftrightarrow \Box\Diamond\neg\psi)$	(Assumption)
(2)	$\Box\varphi \to \Box(\psi \leftrightarrow \Box\Diamond\neg\psi)$	1, $\Box$RM
(3)	$\Box(\psi \leftrightarrow \Box\Diamond\neg\psi) \to (\Box\psi \leftrightarrow \Box\Box\Diamond\neg\psi)$	K, PL
(4)	$\Box\Box\Diamond\neg\psi \leftrightarrow \Box\Diamond\neg\psi$	T, L4
(5)	$(\Box\psi \leftrightarrow \Box\Box\Diamond\neg\psi) \leftrightarrow (\Box\psi \leftrightarrow \Box\Diamond\neg\psi)$	4, Rep
(6)	$\Box\varphi \to (\Box\psi \leftrightarrow \Box\Diamond\neg\psi)$	2, 3, 4, 5
(7)	$\Box\varphi \to \Box(\Box\psi \leftrightarrow \Box\Diamond\neg\psi)$	6, $\Box$RM
(8)	$\Box\varphi \to (\psi \leftrightarrow \Box\psi)$	1, 7, Rep$\Box$
(9)	$\Box\varphi \to \Box(\psi \leftrightarrow \Box\psi)$	8, $\Box$RM
(10)	$\Box(\psi \leftrightarrow \Box\psi) \leftrightarrow \Box(\neg\psi \leftrightarrow \Diamond\neg\psi)$	PL, Df$\Diamond$
(11)	$\Box\varphi \to (\psi \leftrightarrow \Box\neg\psi)$	1, 9, 10, Rep$\Box$
(12)	$\neg\Box\varphi$	11

Inconsistency of ($\Box$Comp$\Box\Diamond\Box$)

Consider next whether the a further variant is consistent in **S4**:

$$(\exists y)(\forall x)\Box(x \in y \leftrightarrow \Box\Diamond\Box\varphi) \qquad (\Box\text{Comp}\Box\Diamond\Box)$$

Once again, we show that it cannot be, using the following instance:

$$(\exists y)\Box(\forall x)(x \in y \leftrightarrow \Box\Diamond\Box\neg x \in x). \qquad (\Box\text{Russell}\Box\Diamond\Box)$$

Theorem. ($\Box$Russell$\Box\Diamond\Box$) is inconsistent in **S4**.

Recall that **S4** proves every instance of the following "reduction law":

$$\Box\Diamond\varphi \leftrightarrow \Box\Diamond\Box\Diamond\varphi. \qquad (\text{Red}\Box\Diamond)$$

The theorem, then, is a consequence of the next proposition.

Proving the Inconsistency of ($\Box$Comp$\Box\Diamond\Box$)

Proposition. If $\vdash \Box\varphi \to (\psi \leftrightarrow \Box\Diamond\Box\neg\psi)$ then $\vdash \neg\Box\varphi$ in **S4**.

Proof.

$$
\begin{array}{rll}
(1) & \Box\varphi \to (\psi \leftrightarrow \Box\Diamond\Box\neg\psi) & \text{(Assumption)} \\
(2) & \Box\varphi \to \Box(\psi \leftrightarrow \Box\Diamond\Box\neg\psi) & 1, \Box\text{RM} \\
(3) & \Box(\psi \leftrightarrow \Box\Diamond\Box\neg\psi) \to (\Box\psi \leftrightarrow \Box\Box\Diamond\Box\neg\psi) & \text{K, PL} \\
(4) & \Box\Box\Diamond\Box\neg\psi \leftrightarrow \Box\Diamond\Box\neg\psi & \text{T, L4} \\
(5) & (\Box\psi \leftrightarrow \Box\Box\Diamond\Box\neg\psi) \leftrightarrow (\Box\psi \leftrightarrow \Box\Diamond\Box\neg\psi) & 4, \text{Rep} \\
(6) & \Box\varphi \to (\Box\psi \leftrightarrow \Box\Diamond\Box\neg\psi) & 2, 3, 4, 5 \\
(7) & \Box\varphi \to \Box(\Box\psi \leftrightarrow \Box\Diamond\Box\neg\psi) & 6, \Box\text{RM} \\
(8) & \Box\varphi \to (\psi \leftrightarrow \Box\psi) & 1, 7, \text{Rep}\Box \\
(9) & \Box\varphi \to \Box(\psi \leftrightarrow \Box\psi) & 8, \Box\text{RM} \\
(10) & \Box(\psi \leftrightarrow \Box\psi) \leftrightarrow \Box(\neg\psi \leftrightarrow \Diamond\neg\psi) & \text{PL, Df}\Diamond \\
(11) & \Box\varphi \to (\psi \leftrightarrow \Box\Diamond\Box\Diamond\neg\psi) & 1, 9, 10, \text{Rep}\Box \\
(12) & \Box\varphi \to (\psi \leftrightarrow \Box\Diamond\neg\psi) & 11, \text{Red}\Box\Diamond \\
(13) & \neg\Box\varphi & 12
\end{array}
$$

Dual Modalities?

S4 has fourteen (well, thirteen) modalities. First there are the seven positive ones:

$$(1)\ \neg\neg,\ (2)\ \Box,\ (3)\ \Diamond,\ (4)\ \Box\Diamond,\ (5)\ \Diamond\Box,\ (6)\ \Box\Diamond\Box,\ (7)\ \Diamond\Box\Diamond$$

If $\bigcirc$ is one of the operators 1, 2, 4, or 6, we know ($\Box$Comp$\bigcirc$) is inconsistent in **S4**. The other operators are the duals $\neg\bigcirc\neg$ (with (1) being self-dual). Consider a principle:

$$(\exists y)\Box(\forall x)(x \in y \leftrightarrow \neg\bigcirc\neg\neg x \in x). \qquad (\Box\text{Russell}\neg\bigcirc\neg)$$

This is an equivalent version:

$$(\exists y)\Box(\forall x)(x \notin y \leftrightarrow \bigcirc\neg x \notin x). \qquad (\Box\text{Russell}\bigcirc\notin)$$

But we can regard $\notin$ as just another binary relation — and then derive a contradiction as before. Thus, *all* positive modalities lead to the inconsistency of ($\Box$Comp$\bigcirc$).

Negative Modalities?

Consider a negative modality $\neg\bigcirc$ where $\bigcirc$ is positive. Can one of these help us?

The following is an instance of ($\Box$Comp$\neg\bigcirc$).

$$(\exists y)\Box(\forall x)(x \in y \leftrightarrow \neg \bigcirc x \in x). \tag{1}$$

But this is equivalent to:

$$(\exists y)\Box(\forall x)(x \in y \leftrightarrow \neg \bigcirc \neg\neg x \in x). \tag{2}$$

And (2) is exactly ($\Box$Russell$\neg \bigcirc \neg$), which, because $\neg \bigcirc \neg$ is positive, we showed inconsistent previously. Thus, (1) is inconsistent in **S4** as well. Thus, not surprisingly, negative modalities are of no help at all.

A Consistency Result

In the system **S4** we have shown that ($\Box$Comp$\bigcirc$) is inconsistent for all 13 modalities. Next, however, we will outline a proof that:

Theorem. *The weaker* (Comp$\Box$) *has a model in* **S5** *also satisfying:*

(Bar)	$\Box\forall x\varphi(x) \leftrightarrow \forall x\Box\varphi(x)$
(Ext)	$(\forall y)(\forall z)\,[(\forall x)\,(x \in y \leftrightarrow x \in z) \rightarrow y = z]$
(Neg)	$(\forall z)(\exists y)(\forall x)\,[x \in y \leftrightarrow \neg(x \in z)]$
(Con)	$(\forall z_1)(\forall z_2)(\exists y)(\forall x)\,[x \in y \leftrightarrow (x \in z_1 \wedge x \in z_2)]$
(Comp$\Diamond$)	$(\exists y)(\forall x)\,[x \in y \leftrightarrow \Diamond\varphi(x)]$
(Equ)	$\forall x\forall y(\Diamond x = y \rightarrow \Box x = y)$
(Mem)	$\forall x\forall y\Diamond x \in y$
(Non)	$\forall x\forall y\Diamond\neg x \in y$

Warning! The above principles in *no way* should be considered as a mathematically motivated version of a "modal naive set theory".

The Model Construction

The idea is based on a very simple "possible worlds" interpretation of the modal system **S5**.

Call a binary relation E on the countably infinite set $\mathbb{N}$ "memberly" if the transformation $m \mapsto \{n \mid n\,E\,m\}$ is a *bijection* between $\mathbb{N}$ and the set of all finite and cofinite subsets of $\mathbb{N}$.

Next, let W, the set of worlds, be the the set of *all* memberly relations. Each E gives the meaning of the membership relation in its own world.

Let $\mathbb{N}!$ be the set of all permutation π of $\mathbb{N}$. For $E \in W$ define:

$$\pi(E) = \{(\pi(n), \pi(m)) \mid n\,E\,m\}.$$

Because $\pi(E)$ is isomorphic to E by π, then π becomes a permutation of the set W as well.

Modal Semantics

We will define *truth* at a possible world for formulae without free variables (but allowing integers in $\mathbb{N}$ as *constants*) as follows:

$$
\begin{aligned}
&E \vDash n = m \text{ iff } n = m\\
&E \vDash n \in m \text{ iff } n\, E\, m\\
&E \vDash \neg\varphi \text{ iff not } E \vDash \varphi\\
&E \vDash \varphi \wedge \psi \text{ iff } E \vDash \varphi \text{ and } E \vDash \psi\\
&E \vDash \varphi \vee \psi \text{ iff } E \vDash \varphi \text{ or } E \vDash \psi\\
&E \vDash (\forall x)\varphi(x) \text{ iff } E \vDash \varphi(n) \text{ for all } n \in \mathbb{N}\\
&E \vDash (\exists x)\varphi(x) \text{ iff } E \vDash \varphi(n) \text{ for some } n \in \mathbb{N}\\
&E \vDash \Box\varphi \text{ iff } F \vDash \varphi \text{ for all } F \in W
\end{aligned}
$$

On the basis of this semantical definition we can prove a key lemma about automorphisms of our model.

Lemma. $E \vDash \varphi(n, m, ...)$ *iff* $\pi(E) \vDash \varphi(\pi(n), \pi(m), ...)$ *if* $\pi \in \mathbb{N}!$

Verifying Comprension

Checking the laws of logic and of **S5** is of no problem. What we need the automorphisms for is proving:

Lemma. *The set* $\{k \,|\, E \vDash \Box\varphi(k, n, m, ...)\}$ *is always finite or cofinite.*

The idea of the proof is that otherwise we could find a k_0 *in* the set and a k_1 *out of* the set *different from* the rest of the constants in the indicated formula. We then take a $\pi \in \mathbb{N}!$ where $\pi(k_0) = k_1$ and π leaves the other constants in the formula *fixed*. By the automorphism principle we then find:

$$E \vDash \Box\varphi(k_0, n, m, ...) \quad \textit{iff} \;\; \pi(E) \vDash \Box\varphi(k_1, n, m, ...) \quad \textit{iff} \\ E \vDash \Box\varphi(k_1, n, m, ...).$$

But this is impossible. And that observation is enough to validate (Comp$\Box$) in the model. Checking the other properties mentioned earlier is equally easy now.

Undecidability

A memberly relation E provides a weak set theory with all finite or cofinite subsets of the domain $\mathbb{N}$. The usual definitions of *unordered and ordered pairs* involve only finite sets. *Bijections* between finite sets, thus, use only finte sets of finite sets ... .

Among all the sets, a *finite set* is characterized by the fact that it is in a one-one correspondence with a set *disjoint from* it. It follows that, in the first-order theory of a memberly relation, we can define what it means for two finite sets to have the *same cardinality*.

Equally obvious is that among finite sets we can define *cartesian product* and *disjoint union*. But this gives us the power of defining in a first-order way the *arithmetic* of finite cardinals.

Therefore, the first-order theory of a memberly relation is shown to be *undecidable*. In our modal theory, the *non-modal* part is just the theory of a memberly relation. This, then, establishes the undecidability of the theory of our modal model.

Any Conclusions?

(I). Very many versions of modal comprehension are inconsistent by proofs analogous to the Russell Paradox.

(II). A plausible version of modal comprehension gives a weak set theory (adequate for finite arithmetic, to be sure) but not supporting a full, transfinite set theory.

(III). It seems hard to argue from this evidence that modalities can do much to save Naive Set Theory!

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